Centrum voor Theoretische Natuurkunde (CTN)
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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Thursday 10-11-11, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 16 parts. The 16 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\,, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\,, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}\,.$$

PROBLEM 1

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{1.1}$$

where Λ is the Lorentz transformation matrix and $(x^\mu)' = \Lambda^\mu{}_\nu x^\nu$.

1.1 Show that the Dirac equation is covariant under Lorentz transformations if

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu = \gamma^\nu$$

and that this implies the equivalent relation

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu\,. \tag{1.2}$$

1.2 Determine the transformation of $\bar{\psi}(x)$ under Lorentz transformations.

1.3 Show that $\bar{\psi}(x)\psi(x)$ is invariant under Lorentz transformations if

$$S^{-1} = \gamma^0 S^\dagger \gamma^0 \,. \tag{1.3}$$

1.4 The interaction term of the photon field and the Dirac field in quantum electrodynamics,

$$\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\,, \tag{1.4}$$

must be invariant under Lorentz transformations. How should the photon field $A_\mu(x)$ transform to achieve this invariance?

PROBLEM 2

The Lagrangian density for the Dirac field is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\,. \tag{2.1}$$

2.1 Define the canonical momentum corresponding to the field ψ, and show that it equals

$$\pi(t,\vec{x}) = i\psi^\dagger(t,\vec{x})\,. \tag{2.2}$$

2.2 The Hamiltonian is defined as

$$H = \int d^3x\,(\pi(t,\vec{x})\partial_0\psi(t,\vec{x}) - \mathcal{L})\,. \tag{2.3}$$

Show that this equals

$$H = -\int d^3x\,\bar{\psi}(i\gamma^k\partial_k - m)\psi\,. \tag{2.4}$$

2.3 The invariance of $\mathcal{L}$ under transformations

$$\psi \to \psi' = e^{-ie\theta}\psi \tag{2.5}$$

gives rise to a current $j^\mu \equiv -e\bar{\psi}\gamma^\mu\psi$. Show that, if the Dirac equation for ψ (and $\bar{\psi}$) holds, j^μ satisfies

$$\partial_\mu j^\mu = 0\,. \tag{2.6}$$

2.4 Show that

$$Q = \int d^3x\, j^0 \tag{2.7}$$

is constant in time if ψ goes sufficiently fast to zero at large $|\vec{x}|$.

2.5 Show that for four arbitrary operators A, B, C, D the following commutation relation holds:

$$[AB, CD] = -C\{D, A\}B + \{C, A\}DB - AC\{D, B\} + A\{C, B\}D\,. \tag{2.8}$$

2.6 The equal-time anticommutation relations for the Dirac field are

$$\{\psi_a(t, \vec{x}), \pi_b(t, \vec{y})\} = i\delta_{ab}\delta^3(\vec{x}-\vec{y})\,,\ \{\psi_a(t, \vec{x}), \psi_b(t, \vec{y})\} = \{\pi_a(t, \vec{x}), \pi_b(t, \vec{y})\} = 0\,. \tag{2.9}$$

Show that under the same conditions as in (2.4)

$$[H, Q] = 0\,. \tag{2.10}$$

PROBLEM 3

The Dirac field $\psi(x)$ satisfies equal-time anti-commutation relations

$$\{\psi_a(x), \psi_b^\dagger(y)\}_{x^0=y^0} = \delta_{ab}\delta^3(\vec{x} - \vec{y})\,, \tag{3.1}$$

where $a, b = 1, \ldots, 4$ are spinor indices. For spinor fields we use the following definition of the time-ordered product

$$T(\psi(x)\bar{\psi}(y)) = \theta(x^0 - y^0)\psi(x)\bar{\psi}(y) - \theta(y^0 - x^0)\bar{\psi}(y)\psi(x)\,. \tag{3.2}$$

3.1 In which sense does this definition differ from the definition of the time-ordered product of two Klein-Gordon fields?

3.2 Using the definition (3.2), show that

$$(i\gamma^\mu\partial_{x^\mu} - m)T(\psi(x)\bar{\psi}(y)) = i\delta^4(x - y)\,. \tag{3.3}$$

3.3 Given that $(\partial_x^2 + m^2)\Delta_{\mathrm{F}}(x - y) = -\delta^4(x - y)$, show that

$$T(\psi(x)\bar{\psi}(y)) = i(i\gamma^\mu\partial_{x^\mu} + m)\Delta_{\mathrm{F}}(x - y) \tag{3.4}$$

satisfies (3.3).

PROBLEM 4

Consider the annihilation of an electron-positron pair into two photons:

$$e^+ + e^- \to \gamma + \gamma\,,$$

with momenta

$e^+ :\ p_1 = (E_1, \vec{p}_1),\ e^- :\ p_2 = (E_2, \vec{p}_2),\ \text{photons}:\ k_1 = (\omega_1, \vec{k}_1),\ k_2 = (\omega_2, \vec{k}_2).$

The process takes place in the laboratory frame:

$$\vec{p}_2 = 0\,.$$

4.1 Why is $\omega_i = |\vec{k}_i|$?

4.2 Express E_1 and $\vec{p}_1$ in terms of the energies and momenta of the two photons.

4.3 The two photons appear under an angle θ in this process: $\vec{k}_1 \cdot \vec{k}_2 = \omega_1\omega_2 \cos\theta$. Express $\cos\theta$ in terms of m, ω_1, ω_2.